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Frédéric Chapoton

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A rooted-trees q-series lifting a one-parameter family of Lie idempotents

F. Chapoton

July 11, 2008

Abstract

We define and study a series indexed by rooted trees and with coefficients in $\mathbb{Q}(q)$. We show that it is related to a family of Lie idempotents. We prove that this series is a q-deformation of a more classical series and that some of its coefficients are Carlitz q-Bernoulli numbers.

1 Introduction

The aim of this article is to introduce and study a series Ω_q indexed by rooted trees, with coefficients that are rational functions of the indeterminate q.

The series Ω_q is in fact an element of the group G_{PL} of formal power series indexed by rooted trees, which is associated to the PreLie operad by a general functorial construction of a group from an operad [Cha02a, vdL03, Cha07b, CL07]. As there is an injective morphism of operads from the PreLie operad to the dendriform operad, there is an injection of groups from G_{PL} to the group G_{Dend}, which is a group of formal power series indexed by planar binary trees. This means that each series indexed by rooted trees can be mapped to a series indexed by planar binary trees, in a non-trivial way.

There is a conjectural description of the image of this injection of groups (see [Cha07a, Corollary 5.4]). This can be stated roughly as the intersection in a bigger space (spanned by permutations) of the dendriform elements with the Lie elements. The inclusion of the image in the intersection is known, but the converse is not.

One starting point of this article was the existence of a one-parameter family of Lie idempotents belonging to the descent algebras of the symmetric groups [DKLT94, KLT97]. As Lie idempotents, these are in particular Lie elements. As elements of the descent algebras, these are also dendriform elements. Therefore, according to the conjecture stated above, they should belong to the image of G_{PL} in G_{Dend}.

Bypassing the conjecture, we prove this by exhibiting an element Ω_q of G_{PL} and then showing that its image is the expected sum of Lie idempotents.

We then obtain several results on Ω_q. First, we prove that the series Ω_q has only simple poles at non-trivial roots of unity and in particular, can be evaluated at $q = 1$. Then we show that Ω_q is a q-deformation of a classical series Ω which is its value at $q = 1$. We also compute the value at $q = 0$ and the appropriate limit value when $q = \infty$.

We then consider the images of Ω_q in some other groups. There are two morphisms of groups from G_{PL} to usual groups of formal power series in one variable. Looking at corollas only, one gets a map from G_{PL} to the group of formal power series with constant term 1 for multiplication. The image of Ω_q is then the generating function of the q-Bernoulli numbers introduced by Carlitz, that appear quite naturally here.

On the other hand, looking at linear trees only, one gets a map from G_{PL} to the composition group of formal power series without constant term. The image of Ω_q is then a q-logarithm.

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2 General setting

We will work over the field $\mathbb{Q}$ of rational numbers and over the field $\mathbb{Q}(q)$ of fractions in the indeterminate q.

We have tried to avoid using operads as much as possible, but this language is needed at some points in this article. The reader may consult [Lod01, Cha07b] as references. The symbol $\circ$ will denote the composition in an operad and the symbols $\flat$ and $\natural$ will serve to note the place where composition is done.

2.1 Pre-Lie algebras

Recall (see for instance [CL01]) that a **pre-Lie algebra** is a vector space V endowed with a bilinear map $\curvearrowleft$ from $V \otimes V$ to V satisfying the following axiom:

$$(x \curvearrowleft y) \curvearrowleft z - x \curvearrowleft (y \curvearrowleft z) = (x \curvearrowleft z) \curvearrowleft y - x \curvearrowleft (z \curvearrowleft y). \tag{1}$$

This is sometimes called a right pre-Lie algebra.

The pre-Lie product $\curvearrowleft$ defines a Lie bracket on V as follows:

$$[x, y] = x \curvearrowleft y - y \curvearrowleft x. \tag{2}$$

One can easily check that the pre-Lie axiom (1) implies the Jacobi identity for the anti-symmetric bracket $[\,,\,]$.

The pre-Lie product $\curvearrowleft$ can also be considered as a right action $\curvearrowleft$ of the associated Lie algebra $(V, [\,,\,])$ on the vector space V. Indeed, one has

$$(x \curvearrowleft y) \curvearrowleft z - (x \curvearrowleft z) \curvearrowleft y = x \curvearrowleft [y, z]. \tag{3}$$

This should not be confused with the adjoint action of a Lie algebra on itself.

2.2 Free pre-Lie algebras

The free pre-Lie algebras have a simple description using rooted trees. Let us recall briefly this description and other properties. Details can be found in [CL01].

A **rooted tree** is a finite, connected and simply connected graph, together with a distinguished vertex called the root. We will picture rooted trees with their root at the bottom and orient (implicitly) the edges towards the root.

There are two distinguished kinds of rooted trees: corollas (every vertex other than the root is linked to the root by an edge) and linear trees (at every vertex, there is at most one incoming edge), see Fig. 1. A **forest of rooted trees** is a finite graph whose connected components are rooted trees.

The free pre-Lie algebra $\mathsf{PL}(S)$ on a set S has a basis indexed by rooted trees decorated by S, *i.e.* rooted trees together with a map from their set of vertices to S.

The pre-Lie product $T \curvearrowleft T'$ of a tree T' on another one T is given by the sum of all possible trees obtained from the disjoint union of T and T' by adding an edge from the root of T' to one of the vertices of T (the root of the resulting tree is the root of T).

In particular, we will denote by PL the free pre-Lie algebra on one generator. This is the graded vector space $\mathsf{PL} = \oplus_{n\geq 1}\mathsf{PL}_n$ spanned by unlabeled rooted trees, where the degree of a tree T is the number $\#T$ of its vertices. The pre-Lie product obviously preserves this grading. We will denote by $\star$ the associative product in the universal enveloping algebra $U(\mathsf{PL})$ of the Lie algebra PL.

There exists a unique isomorphism ψ of graded right PL-modules between the free right $U(\mathsf{PL})$-module on one generator g of degree 1 and the PL-module $(\mathsf{PL}, \curvearrowleft)$ such that ψ maps the generator g to $\bullet$, the unique rooted tree with one vertex.

This means that there is a commutative diagram as follows:

$$\begin{tikzcd} \mathbb{Q}g \otimes U(\mathsf{PL}) \otimes \mathsf{PL} \arrow[r, "\psi\otimes\mathrm{Id}"] \arrow[d, "\mathrm{Id}\otimes\star"] & \mathsf{PL} \otimes \mathsf{PL} \arrow[d, "\curvearrowleft"] \\ \mathbb{Q}g \otimes U(\mathsf{PL}) \arrow[r, "\psi"] & \mathsf{PL} \end{tikzcd} \quad (4)$$

One can use the bijection ψ and the canonical basis of PL to get a canonical basis of the enveloping algebra $U(\mathsf{PL})$ indexed by forests of rooted trees. The degree of a forest F is the number of its vertices $\#F$. The image of the usual inclusion of PL in $U(\mathsf{PL})$ is the subspace spanned by rooted trees. In this basis of $U(\mathsf{PL})$, there is a nice combinatorial description of the associative product $\star$. Let F and F' be forests in $U(\mathsf{PL})$. The product $F \star F'$ is the sum of all possible forests, obtained from the disjoint union of F and F' by the addition of some edges (possibly none), each of these new edges going from some root of F' to some vertex of F.

There is a canonical projection π from $U(\mathsf{PL})$ to PL, defined using the canonical basis of $U(\mathsf{PL})$ by projection on the subspace spanned by rooted trees, annihilating the empty forest and all forests that are not trees.

Lemma 2.1 *Let F be a forest in $U(\mathsf{PL})$ and T be a rooted tree in PL. Then one has $\pi(F \star T) = \pi(F) \curvearrowleft T$.*

Proof. If F is not a tree, then each term of $F \star T$ is not a tree, therefore both sides vanish. If $F = \pi(F)$ is a tree, then $F \star T$ is the sum of $\pi(F) \curvearrowleft T$ with the disjoint union of F and T. Therefore $\pi(F \star T) = \pi(F) \curvearrowleft T$. ■

Lemma 2.2 *For all $n \geq 1$, the maps $T \mapsto \bullet \curvearrowleft T$ and $T \mapsto T \curvearrowleft \bullet$ are injective from PL_n to PL_{n+1}.*

Proof. This is obvious for the first map, which is even an injection on the set of rooted trees. For the second map, this follows from the fact that enveloping algebras are integral domains, by restriction of the commutative diagram (4). ∎

In the sequel, we will always work in the completed vector space $\widehat{\mathsf{PL}} = \prod_{n\geq 1} \mathsf{PL}_n$ and with its completed enveloping algebra $\widehat{U}(\mathsf{PL})$. All the results above are still true in this setting.

There is a group associated to each operad, see [Cha02a, vdL03, Cha07b, CL07]. We will need the group G_{PL} associated to the PreLie operad. Its elements are the elements of $\widehat{\mathsf{PL}}$ whose homogeneous component of degree 1 is $\bullet$. Product in G_{PL} is defined using the composition of the PreLie operad and $\bullet$ is the unit in G_{PL}. This group is contained in the bigger monoid $\widehat{\mathsf{PL}}$, on which it therefore acts on the right and on the left. The right action respects all the operations on $\widehat{\mathsf{PL}}$ induced by the product $\curvearrowleft$, including the product and the action of $\widehat{U}(\mathsf{PL})$.

Let us now introduce a special element of G_{PL}, for later use. Let $\exp^* \in G_{\mathsf{PL}}$ be

$$\exp^* = \bullet \curvearrowleft \left((\exp(\bullet) - 1)/\bullet \right). \tag{5}$$

The series $\exp^*$ is very classical, and its coefficients are known as the Connes-Moscovici coefficients (see [Cha02a]).

Let us consider the left action of $\exp^*$ on $\widehat{\mathsf{PL}}$. Let T be an element of $\widehat{\mathsf{PL}}$. Then $\exp^*(T)$ of $\widehat{\mathsf{PL}}$ is defined by

$$\exp^*(T) = \sum_{n\geq 1} \frac{1}{n!} ((T \curvearrowleft T) \curvearrowleft \dots) \curvearrowleft T, \tag{6}$$

where there are n copies of T in the n^{th} term. As $\exp^*$ belongs to the group G_{PL}, the map $\exp^*$ defines a bijection from $\widehat{\mathsf{PL}}$ to itself.

Let us now relate the usual exponential map exp to the map $\exp^*$.

Let T be an element of $\widehat{\mathsf{PL}}$. Let $\exp(T)$ be the exponential of T in $\widehat{U}(\mathsf{PL})$ (which is defined by the usual series and using the $\star$ product). The map exp defines a bijection from $\widehat{\mathsf{PL}}$ to the set of group-like elements of $\widehat{U}(\mathsf{PL})$.

Therefore, the composite map $\exp^* \circ \exp^{-1}$ is a bijection from the set of group-like elements in $\widehat{U}(\mathsf{PL})$ to $\widehat{\mathsf{PL}}$. Let us show that this composite map is just a restriction of the canonical projection π.

Proposition 2.3 *Let T be an element of $\widehat{\mathsf{PL}}$. One has $\pi(\exp(T)) = \exp^*(T)$.*

Proof. Let F be in $\widehat{U}(\mathsf{PL})$. From Lemma 2.1 above, one knows that $\pi(F \star T)$ is exactly $\pi(F) \curvearrowleft T$. This implies that

$$\pi(T^{\star n}) = ((T \curvearrowleft T) \dots) \curvearrowleft T, \tag{7}$$

for all $n \geq 1$, hence the result. ∎

3 The classical case

Let us start by recalling the definition of a classical element Ω of $\widehat{\mathsf{PL}}$ with rational coefficients. It was considered under the name of $\log^*$ in [Cha02a] and has been since studied in [Mur06, WZ03, EFM08, CEFM08].

Proposition 3.1 *There is a unique solution* Ω *in* $\widehat{\mathsf{PL}}_{\mathbb{Q}}$ *to the equation*

$$\bullet \curvearrowleft \left(\frac{\Omega}{\exp(\Omega) - 1} \right) = \Omega, \tag{8}$$

where $\frac{\Omega}{\exp(\Omega)-1}$ *is in the completed enveloping algebra* $\widehat{U}(\mathsf{PL})$.

Proof. Let us write $\Omega = \sum_{n\geq 1} \Omega_n$ where each Ω_n is homogeneous of degree n.

Recall the Taylor expansion

$$\frac{x}{\exp(x) - 1} = \sum_{k\geq 0} \frac{B_k}{k!} x^k, \tag{9}$$

where the B_k are the Bernoulli numbers.

Then the homogeneous component of degree n of equation (8) is

$$\Omega_n = \sum_{k\geq 0} \frac{B_k}{k!} \sum_{\substack{m_1\geq 1,\dots,m_k\geq 1 \\ m_1+\dots+m_k=n-1}} ((\bullet \curvearrowleft \Omega_{m_k}) \dots) \curvearrowleft \Omega_{m_1}. \tag{10}$$

This gives a recursive definition of Ω_n, which implies the existence and uniqueness of Ω. ∎

Remark: one can use equation (10) to compute Ω up to order n in a $O(n^3)$ number of pre-Lie operations.

As the element $\frac{\Omega}{\exp(\Omega)-1}$ is invertible in the completed enveloping algebra, equation (8) is also equivalent to the following equation:

$$\Omega \curvearrowleft \left(\frac{\exp(\Omega) - 1}{\Omega} \right) = \bullet. \tag{11}$$

One can interpret equation (11) as follows.

Proposition 3.2 *The series* Ω *is the inverse of* $\exp^*$ *in the group* G_{PL}.

Proof. By right action by the inverse Ω^{-1} of Ω in G_{PL} on (11), one shows that Ω^{-1} satisfies the same equation (5) as $\exp^*$. ∎

There is another equation for Ω.

Proposition 3.3 *The series* Ω *is the unique non-zero solution in* $\widehat{\mathsf{PL}}_{\mathbb{Q}}$ *to the equation*

$$\Omega \curvearrowleft (\exp(\Omega) - 1) = \bullet \curvearrowleft \Omega, \tag{12}$$

where $\exp(\Omega) - 1$ *is in the completed enveloping algebra* $\widehat{U}(\mathsf{PL})$.

Proof. First, by right action on (11) by Ω, one can see that the unique solution Ω of (8) is indeed a solution of (12).

Let us now prove uniqueness of a non-zero solution. Let Ω be any solution of (12). Let us write $\Omega = \sum_{n\geq 1} \Omega_n$ where each Ω_n is homogeneous of degree n.

Then the homogeneous component of degree n of equation (12) is

$$\bullet \curvearrowleft \Omega_{n-1} = \sum_{k\geq 1} \frac{1}{k!} \sum_{\substack{m_1\geq 1,\dots,m_k\geq 1,\ell\geq 1\\ m_1+\dots+m_k+\ell=n}} ((\Omega_\ell \curvearrowleft \Omega_{m_k})\dots) \curvearrowleft \Omega_{m_1}. \tag{13}$$

If $n = 2$, this implies that Ω_1 is either 0 or $\bullet$.

Assume now that Ω is not zero. Let d be the degree of the first non-zero homogeneous component Ω_d of Ω. Assume that $d > 1$. Then the equation (13) in degree $d+1$, together with Lemma 2.2, gives that $\Omega_d = 0$, a contradiction. Therefore necessarily, one has $d = 1$ and $\Omega_1 = \bullet$.

Let us look at the homogeneous component (13) in degree $n+1 \geq 2$. The only terms involving Ω_n are $\bullet \curvearrowleft \Omega_n$ in the left-hand side and $\Omega_1 \curvearrowleft \Omega_n$, $\Omega_n \curvearrowleft \Omega_1$ in the right hand-side. As $\Omega_1 = \bullet$, two of them cancel out and one gets a recursive expression of $\Omega_n \curvearrowleft \bullet$ in terms of some Ω_j for $j < n$.

Using Lemma 2.2, this provides a recursive description of Ω (that may or may not possess a solution) and proves its uniqueness. ■

The exponential of Ω has a simple shape.

Proposition 3.4 *In the enveloping algebra $\widehat{U}(\mathsf{PL})$, one has*

$$\exp(\Omega) = \sum_{n\geq 0} \frac{1}{n!} \bullet\bullet\dots\bullet, \tag{14}$$

where, in the n^{th} term, the forest has n nodes.

Proof. This is an equation for the exponential $\exp(\Omega)$ of the element Ω in the Lie algebra $\widehat{\mathsf{PL}}$. By Proposition 2.3, it is enough to prove that

$$\exp^*(\Omega) = \bullet, \tag{15}$$

because the image by π of the right-hand side of (14) is $\bullet$.

But this amounts to say that $\exp^*$ is the inverse of Ω in the group G_{PL}. This is nothing else that Proposition 3.2. ■

It follows that

$$\Omega \curvearrowleft (\exp(\Omega) - 1) = \sum_{n\geq 2} \frac{1}{(n-1)!} \mathtt{Crl}_n^\natural \circ_\natural \Omega, \tag{16}$$

where $\mathtt{Crl}_n^\natural$ is the corolla with $n-1$ leaves and with root labeled by $\natural$, see Fig. 1.

Proposition 3.5 *The series Ω is the unique non-zero solution in $\widehat{\mathsf{PL}_{\mathbb{Q}}}$ to the equation*

$$\sum_{n\geq 2} \frac{1}{(n-1)!} \mathtt{Crl}_n^\natural \circ_\natural \Omega = \bullet \curvearrowleft \Omega, \tag{17}$$

where $\sum_{n\geq 2} \frac{1}{(n-1)!} \mathtt{Crl}_n^\natural$ is in $\widehat{\mathsf{PL}}$.

Proof. This follows from Eq. (16) and Prop. 3.3. ■

4 The quantum case

We will introduce now an element Ω_q in $\widehat{\mathsf{PL}}$ with coefficients in $\mathbb{Q}(q)$. We will show later that this is a q-deformation of Ω.

If $A = \sum_{n\geq 1} A_n$ is an element of $\widehat{\mathsf{PL}}$, let $A[q]$ be the q-shift of A defined by

$$A[q] = \sum_{n\geq 1} q^n A_n. \tag{18}$$

Proposition 4.1 *There exists a unique solution Ω_q in $\mathsf{PL}_{\mathbb{Q}(q)}$ to the equation*

$$\Omega_q[q] \curvearrowleft (\exp(\Omega) - 1) + \Omega_q[q] - \Omega_q = \bullet \curvearrowleft \Omega_q + (q-1)\bullet. \tag{19}$$

Moreover, the series Ω_q has coefficients in the ring of fractions with poles only at roots of unity.

Proof. Let us write $\Omega_q = \sum_{n\geq 1} \Omega_{q,n}$ where each $\Omega_{q,n}$ is homogeneous of degree n. The homogeneous component of degree 1 of (19) implies that $\Omega_{q,1} = \bullet$.

Then for $n \geq 2$, the homogeneous component of degree n of equation (19) is

$$(q^n - 1)\Omega_{q,n} = \bullet \curvearrowleft \Omega_{q,n-1} - \sum_{k\geq 1} \frac{1}{k!} \sum_{\substack{m_1\geq 1,\dots,m_k\geq 1,\ell\geq 1\\ m_1+\dots+m_k+\ell=n}} q^\ell ((\Omega_{q,\ell} \curvearrowleft \Omega_{m_k})\dots) \curvearrowleft \Omega_{m_1}. \tag{20}$$

This provides an explicit recursion for $\Omega_{q,n}$ in terms of $\Omega_{q,j}$ and Ω_j for $j < n$. This gives existence and uniqueness and also implies that Ω_q has coefficients with poles only at roots of unity. ■

On can reformulate the equation for Ω_q.

Proposition 4.2 *The series Ω_q is the unique solution in $\widehat{\mathsf{PL}}_{\mathbb{Q}(q)}$ to the equation*

$$\sum_{n\geq 1} \frac{1}{(n-1)!} \mathtt{Crl}^\natural_n \circ_\natural \Omega_q[q] - \Omega_q = \bullet \curvearrowleft \Omega_q + (q-1)\bullet. \tag{21}$$

Proof. This follows from Eq. (16) and Prop. 4.1. ■

Let $\mathtt{Frk}^\natural_{\ell,n}$ be the rooted tree with a linear trunk of ℓ vertices, a vertex $\natural$ on top of this trunk and a corolla with n leaves on top of the vertex $\natural$, see Fig. 1. We will call this a **fork**. One has $\mathtt{Frk}_{\ell,n} = \mathtt{Lnr}^\flat_{\ell+1} \circ_\flat \mathtt{Crl}^\natural_{n+1}$.

Proposition 4.3 *The series Ω_q is the unique solution in $\widehat{\mathsf{PL}}_{\mathbb{Q}(q)}$ to the equation*

$$\Omega_q = \sum_{\ell\geq 0}\sum_{n\geq 0} \frac{(-1)^\ell}{n!} \mathtt{Frk}^\natural_{\ell,n} \circ_\natural \Omega_q[q] + (1-q)\sum_{\ell\geq 1}(-1)^{\ell-1}\mathtt{Lnr}_\ell. \tag{22}$$

Proof. Let us compute the right-hand side of Eq. (22), using Eq. (21) for Ω_q, written as

$$\Omega_q + \bullet \curvearrowleft \Omega_q + (q-1)\bullet = \sum_{n\geq 1} \frac{1}{(n-1)!} \mathtt{Crl}^\natural_n \circ_\natural \Omega_q[q]. \tag{23}$$

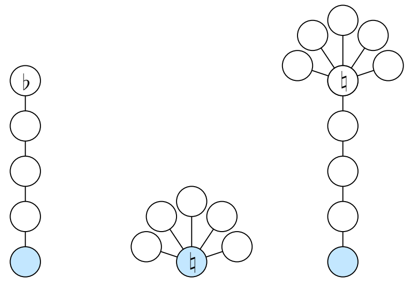


Figure 1: Rooted trees: $\mathtt{Lnr}^{\flat}_5$, $\mathtt{Crl}^{\natural}_6$ and $\mathtt{Frk}^{\natural}_{4,5} = \mathtt{Lnr}^{\flat}_5 \circ_{\flat} \mathtt{Crl}^{\natural}_6$.

One gets

$$\sum_{\ell \geq 1} (-1)^{\ell-1} \mathtt{Lnr}^{\flat}_{\ell} \circ_{\flat} (\Omega_q + \bullet \curvearrowleft \Omega_q + (q-1)\bullet) + (1-q) \sum_{\ell \geq 1} (-1)^{\ell-1} \mathtt{Lnr}_{\ell}. \quad (24)$$

As $\mathtt{Lnr}^{\flat}_{\ell} \circ_{\natural} \bullet = \mathtt{Lnr}_{\ell}$ and $\mathtt{Lnr}^{\flat}_{\ell} \circ_{\flat} (\bullet \curvearrowleft \Omega_q) = \mathtt{Lnr}^{\flat}_{\ell+1} \circ_{\flat} \Omega_q$, the two right-most terms cancels, and the sum simplifies to

$$\sum_{\ell \geq 1} (-1)^{\ell-1} \mathtt{Lnr}^{\flat}_{\ell} \circ_{\flat} \Omega_q - \sum_{\ell \geq 2} (-1)^{\ell-1} \mathtt{Lnr}^{\flat}_{\ell} \circ_{\flat} \Omega_q, \quad (25)$$

which is just Ω_q. This proves that Ω_q does satisfy Eq. (22).

It is then easy to see that (22) has only one solution in $\widehat{\mathsf{PL}}_{\mathbb{Q}(q)}$ by rewriting it as a recursion for the homogeneous components $\Omega_{q,n}$. ■

5 Image in the free dendriform algebra

We describe in this section the image of Ω_q by the usual morphism from the free pre-Lie algebra to the free dendriform algebra. We show that this image is related to a family of Lie idempotents in the descent algebras of the symmetric groups. One deduces from that a nice explicit formula, that will be used later to get arithmetic information on Ω_q.

5.1 Dendriform algebra

Recall that a **dendriform algebra** (notion due to Loday, see [Lod01]) is a vector space V endowed with two bilinear maps $\succ$ and $\prec$ from $V \otimes V$ to V satisfying the following axioms:

$$x \prec (y \prec z) + x \prec (y \succ z) = (x \prec y) \prec z, \quad (26)$$
$$x \succ (y \prec z) = (x \succ y) \prec z, \quad (27)$$
$$x \succ (y \succ z) = (x \succ y) \succ z + (x \prec y) \succ z. \quad (28)$$

Any dendriform algebra has the structure of a pre-Lie algebra given by

$$x \curvearrowleft y = y \succ x - x \prec y. \quad (29)$$

Any dendriform algebra has the structure of an associative algebra given by

$$x * y = x \succ y + x \prec y. \tag{30}$$

Remark: Eq. (27) means that one can safely forget some parentheses. Eq. (26) and (28) can be rewritten as

$$x \prec (y * z) = (x \prec y) \prec z, \tag{31}$$

$$x \succ (y \succ z) = (x * y) \succ z. \tag{32}$$

Let $\mathrm{Dend}(S)$ be the free dendriform algebra over a set S. This has an explicit basis indexed by **planar binary trees** with vertices decorated by S. For an example of a planar binary tree, see Fig. 2. In particular, the free dendriform algebra on one generator, denoted by Dend, has a basis indexed by planar binary trees. This is a graded vector space, the degree $\#t$ of a planar binary tree t being the number of its inner vertices.

There is a unique morphism φ of pre-Lie algebras from PL to Dend that maps the rooted tree $\bullet$ to the planar binary tree $\curlyvee$. This extends uniquely to a continuous morphism φ from $\widehat{\mathsf{PL}}$ to the completion $\widehat{\mathrm{Dend}}$ of Dend.

Remark: with some care, one can add a unit 1 to the free dendriform algebra Dend. Then one has $1 * x = 1 \succ x = x = x \prec 1 = x * 1$, but one has to pay attention to never write neither $1 \prec x$ nor $x \succ 1$. We will use this convention in the sequel.

There are two kinds of special planar binary trees: the left combs and the right combs. They can be defined as follows. Let $L = \sum_{n \geq 1} L_n$ be the unique solution in $\widehat{\mathrm{Dend}}$ to the equation

$$L = \curlyvee + L \succ \curlyvee = (1 + L) \succ \curlyvee, \tag{33}$$

and let $R = \sum_{n \geq 1} R_n$ be the unique solution in $\widehat{\mathrm{Dend}}$ to

$$R = \curlyvee + \curlyvee \prec R = \curlyvee \prec (1 + R). \tag{34}$$

Then L_n is called the left comb with n vertices and R_n be the right comb with n vertices.

If $A = \sum_{n \geq 1} A_n$ is an element of $\widehat{\mathsf{PL}}$ or $\widehat{\mathrm{Dend}}$, the **suspension** of A is $\widetilde{A} = \sum_{n \geq 1} (-1)^{n-1} A_n$.

Proposition 5.1 *The inverse of $1 + R$ with respect to the $*$ product is $1 - \widetilde{L}$.*

Proof. One has $\widetilde{L} = \curlyvee - \widetilde{L} \succ \curlyvee$. Let us compute

$$(1 - \widetilde{L}) * (1 + R) = 1 + R - \widetilde{L} * (1 + R). \tag{35}$$

By the definition of $*$ and the convention on the unit 1, this is

$$1 + R - \widetilde{L} \prec (1 + R) - \widetilde{L} \succ R \tag{36}$$

By Eq. (33), this becomes

$$1 + R - \curlyvee \prec (1 + R) + \widetilde{L} \succ \curlyvee \prec (1 + R) - \widetilde{L} \succ R. \tag{37}$$

The last two terms cancel by Eq. (34) and one gets

$$1 + R - \curlyvee \prec (1 + R), \tag{38}$$

which is just 1, again by Eq. (34). ∎

5.2 Equation for the dendriform image of Ω_q

Let us define a series $E = \sum_{n\geq 1} nL_n$ in $\widehat{\mathrm{Dend}}$. One can easily show that

$$E = L + E \succ \curlyvee. \tag{39}$$

Lemma 5.2 *The series $B^\flat = \varphi(\sum_{n\geq 1} \mathtt{Lnr}_n^\flat)$ satisfies*

$$B^\flat = \curlyvee^\flat + B^\flat \succ \curlyvee - \curlyvee \prec B^\flat. \tag{40}$$

Proof. This comes from a similar equation in PL. Let $\mathtt{Lnr}^\flat = \sum_{n\geq 1} \mathtt{Lnr}_n^\flat$. Then

$$\mathtt{Lnr}^\flat = \bullet^\flat + \bullet \curvearrowleft \mathtt{Lnr}^\flat, \tag{41}$$

as one can easily check. ■

These relations can be taken as definitions of the elements E and $B^\flat$ of $\widehat{\mathrm{Dend}}$. One can forget the marking $\flat$ in $B^\flat$ to define a series B.

Proposition 5.3 *The series $B = \varphi(\sum_{n\geq 1} \mathtt{Lnr}_n)$ satisfies*

$$E = (1 + L) * B. \tag{42}$$

Proof. One has to show that $E = (1 + L) * B$. It is enough to prove that $(1 + L) * B$ does satisfy the defining relation (39) of E.

One computes, using Eq. (40) for B,

$$(1 + L) * B = (1 + L) * (\curlyvee + B \succ \curlyvee - \curlyvee \prec B). \tag{43}$$

Expanding the $*$ product, this is

$$(1+L) \succ \curlyvee + L \prec \curlyvee + (1+L) \succ (B \succ \curlyvee - \curlyvee \prec B) + L \prec (B \succ \curlyvee - \curlyvee \prec B). \tag{44}$$

Using Eq. (33) and the dendriform axioms, this becomes

$$L + L \prec \curlyvee + ((1 + L) * B) \succ \curlyvee - (1 + L) \succ \curlyvee \prec B + L \prec (B \succ \curlyvee - \curlyvee \prec B). \tag{45}$$

Using Eq. (33) again, one gets

$$L + ((1 + L) * B) \succ \curlyvee + L \prec (\curlyvee - B + B \succ \curlyvee - \curlyvee \prec B). \tag{46}$$

This simplifies, by Eq. (40) for B, to

$$L + ((1 + L) * B) \succ \curlyvee, \tag{47}$$

as expected. ■

Lemma 5.4 *The image of $\sum_{n\geq 1} \frac{1}{(n-1)!} \mathtt{Crl}_n^\natural$ by φ is*

$$(1 + R) \succ \curlyvee^\natural \prec (1 - \widetilde{L}). \tag{48}$$

Proof. This was proved in [Ron01, Ron00, Cha02b]. ■

Proposition 5.5 *The image of* $\sum_{\ell \geq 0} \sum_{n \geq 0} \frac{(-1)^\ell}{n!} \mathtt{Frk}^{\natural}_{\ell,n}$ *by* φ *is*

$$(1+R) * \curlyvee^{\natural} * (1 - \widetilde{L}), \tag{49}$$

where $\curlyvee^{\natural}$ *is the planar binary tree* $\curlyvee$ *with vertex labeled by* $\natural$.

Proof. Let $D = (1+R) * \curlyvee^{\natural} * (1-\widetilde{L})$. Let us first show that

$$D = (1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L}) + \curlyvee \prec D - D \succ \curlyvee. \tag{50}$$

Expanding the $*$ product, one computes

$$D = ((1+R) * \curlyvee^{\natural}) \prec (1-\widetilde{L}) - ((1+R) * \curlyvee^{\natural}) \succ \widetilde{L}. \tag{51}$$

Then one gets, by expanding again,

$$(1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L}) + (R \prec \curlyvee^{\natural}) \prec (1-\widetilde{L}) - ((1+R) * \curlyvee^{\natural}) \succ \widetilde{L}. \tag{52}$$

Using the dendriform axioms, this is

$$(1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L}) + R \prec (\curlyvee^{\natural} * (1-\widetilde{L})) - ((1+R) * \curlyvee^{\natural}) \succ \widetilde{L}. \tag{53}$$

Then by Eq. (33) and (34), this can be rewritten

$$\begin{aligned}(1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L}) + (\curlyvee \prec (1+R)) \prec (\curlyvee^{\natural} * (1-\widetilde{L}))& \\ - ((1+R) * \curlyvee^{\natural}) \succ ((1-\widetilde{L}) \succ \curlyvee).&\end{aligned} \tag{54}$$

One gets, using the dendriform axioms,

$$(1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L}) + \curlyvee \prec ((1+R) * \curlyvee^{\natural} * (1-\widetilde{L})) - ((1+R) * \curlyvee^{\natural} * (1-\widetilde{L})) \succ \curlyvee. \tag{55}$$

This proves the equation (50) for D.

Let us show now that $D' = (\sum_{\ell \geq 1} (-1)^{\ell-1} \varphi(\mathtt{Lnr}^{\flat}_{\ell})) \circ_\flat (\sum_{n \geq 1} \frac{1}{(n-1)!} \varphi(\mathtt{Crl}^{\natural}_n))$ does satisfy the same equation as D.

By Lemma 5.4, one has

$$D' = \widetilde{B}^\flat \circ \flat((1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L})). \tag{56}$$

By Lemma 5.2, one has $\widetilde{B}^\flat = \curlyvee^\flat - \widetilde{B}^\flat \succ \curlyvee + \curlyvee \prec \widetilde{B}^\flat$, hence

$$D' = (1+R) \succ \curlyvee^{\natural} \prec (1-\widetilde{L}) + \curlyvee \prec D' - D' \succ \curlyvee. \tag{57}$$

By uniqueness of the solution D of Eq. (50), one has $D = D'$, *i.e.*

$$(1+R) * \curlyvee^{\natural} * (1-\widetilde{L}) = (\sum_{\ell \geq 1} (-1)^{\ell-1} \varphi(\mathtt{Lnr}^{\flat}_{\ell})) \circ_\flat (\sum_{n \geq 1} \frac{1}{(n-1)!} \varphi(\mathtt{Crl}^{\natural}_n)). \tag{58}$$

Therefore

$$(1+R) * \curlyvee^{\natural} * (1-\widetilde{L}) = \varphi\left((\sum_{\ell \geq 1} (-1)^{\ell-1} \mathtt{Lnr}^{\flat}_{\ell}) \circ_\flat (\sum_{n \geq 1} \frac{1}{(n-1)!} \mathtt{Crl}^{\natural}_n) \right), \tag{59}$$

which is exactly the expected image by φ of a sum over forks. ∎

One can now deduce a useful functional equation for the image of Ω_q by φ, using only the associative product $*$ of Dend.

Proposition 5.6 *The series $\varphi(\Omega_q)$ is the unique solution in $\widehat{\mathrm{Dend}}$ of*

$$\varphi(\Omega_q) = (1 - \widetilde{L})^{-1} * \varphi(\Omega_q)[q] * (1 - \widetilde{L}) + (1 - q) \sum_{\ell \geq 1} (-1)^{\ell-1} \varphi(\mathtt{Lnr}_\ell). \quad (60)$$

Proof. Let us start from Eq. (22). By Prop. 5.5, we know the image by φ of the sum over forks. One gets

$$\varphi(\Omega_q) = ((1 + R) * \curlyvee^{\natural} * (1 - \widetilde{L})) \circ_{\natural} \varphi(\Omega_q)[q] + (1 - q) \sum_{\ell \geq 1} (-1)^{\ell-1} \varphi(\mathtt{Lnr}_\ell). \quad (61)$$

Then one can used Prop. 5.1 to replace $1 + R$ by the inverse of $1 - \widetilde{L}$. ■

5.3 Explicit formula

We will prove in this section that the image of Ω_q by φ coincides (in some sense) with a known family of Lie idempotents, and has an explicit description using q-binomial coefficients, descents and major indices of planar binary trees. To obtain this description, we use a result on noncommutative symmetric functions. We refer to the articles [GKL$^+$95, KLT97, DKKT97] for background on this subject. We will use the notations of [KLT97].

The algebra **Sym** of noncommutative symmetric function is the free unital associative algebra on generators $S_1, S_2, \dots$. It is a graded algebra (with S_i of degree i), with a basis $(S_I)_I$ indexed by compositions. There is another basis $(R_I)_I$ obtained from the basis $(S_I)_I$ by Möbius inversion on compositions ordered by refinement. By convention, S_0 is the unit of **Sym**.

As **Sym** is free, there is a unique morphism θ from **Sym** to Dend which maps S_i to the left comb L_i for each $i \geq 0$, with the convention that L_0 is the unit of Dend.

One can check that θ is the usual morphism from **Sym** to Dend, considered for instance in [HNT05, §4.8] and [LR98].

In **Sym**, there are elements Ψ_i for $i \geq 1$, uniquely defined by the conditions

$$nS_n = \sum_{i=0}^{n-1} S_i * \Psi_{n-i}, \quad (62)$$

for all $n \geq 1$.

Proposition 5.7 *The image of Ψ_i by θ is $\varphi(\mathtt{Lnr}_i)$.*

Proof. This is a corollary of Prop. 5.3. Indeed, one has

$$1 + L = \sum_{n \geq 0} \theta(S_n) \quad \text{and} \quad E = \sum_{n \geq 1} n\theta(S_n). \quad (63)$$

Therefore

$$B = \sum_{n \geq 1} \theta(\Psi_n). \quad (64)$$

■

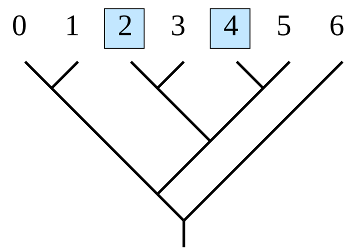


Figure 2: A planar binary tree t with descents at 2 and 4.

We need to introduce the following notations.

The leaves of a planar binary tree with n vertices are labelled from 0 to n from left to right. The leaves with labels different from 0 and n are called **inner leaves**. A **descent** in a planar binary tree t is the label of an $\backslash$-oriented inner leaf. The descent set $D(t)$ of t is the set of its descents.

The number of descents of a planar binary tree t will be denoted $d(t)$. It satisfies $0 \leq d(t) \leq n-1$ for a tree t of degree n.

The **major index** $\mathrm{maj}(t)$ of t is the sum of its descents. For example, Fig. 2 displays a planar binary tree with descent set $\{2,4\}$ and major index $2+4=6$.

Let us recall that the descent set $D(I)$ corresponding to a composition $I=(i_1,\ldots,i_k)$ of n is the set $\{i_1, i_1+i_2, \ldots, i_1+\cdots+i_k\}$.

Proposition 5.8 *The image by θ of R_I is the sum*

$$\sum_{\substack{\#t=n\\ D(t)=D(I)}} t \tag{65}$$

of all planar binary trees with n vertices and descent set $D(I)$.

Proof. This is a well-known property of the injection of **Sym** in Dend. ■

In [KLT97], elements $\Psi_n(\frac{A}{1-q})$, for $n \geq 1$, are defined by some "change of alphabet" applied to the elements Ψ_n. According to the proof of [KLT97, Theorem 6.11], they are characterized by

$$\sum_{n\geq 1} \Psi_n(\frac{A}{1-q}) = \left(\sum_{n\geq 0} S_n\right)^{-1} \left(\sum_{n\geq 1} q^n \Psi_n(\frac{A}{1-q})\right) \left(\sum_{n\geq 0} S_n\right) + \sum_{n\geq 1} \Psi_n. \tag{66}$$

There is a classical isomorphism α from **Sym** to the direct sum of all descent algebras of symmetric groups. By this morphism, up to a multiplicative constant, each $\Psi_n(\frac{A}{1-q})$ is mapped to a Lie idempotent with coefficients in $\mathbb{Q}(q)$ in the descent algebra of the n^{th} symmetric group.

We can now state the precise relation between Ω_q and these Lie idempotents.

Proposition 5.9 *The image of $(1-q)\Psi(\frac{A}{1-q})$ by θ is $\varphi(\widetilde{\Omega_q})$.*

Proof. Indeed, by Proposition 5.6, one has

$$\sum_{n\geq 1}\varphi(\widetilde{\Omega_q}) = (1+L)^{-1} * (\varphi(\widetilde{\Omega_q})[q]) * (1+L) + (1-q)\sum_{n\geq 1}\varphi(\mathtt{Lnr}_n). \qquad (67)$$

Then using Prop. 5.7 and Eq. (66), one gets that $\theta((1-q)\Psi(\frac{A}{1-q}))$ and $\varphi(\widetilde{\Omega_q})$ satisfy the same equation, hence they are equal. ∎

Let $\Omega_{q,n}$ be the homogeneous component of degree n of Ω_q.

Proposition 5.10 *One has*

$$\varphi(\Omega_{q,n}) = \frac{(-1)^{n-1}}{[n]_q}\sum_{\#t=n}(-1)^{d(t)}\begin{bmatrix} n-1 \\ d(t)\end{bmatrix}_q^{-1} q^{\mathrm{maj}(t)-\binom{d(t)+1}{2}}\, t. \qquad (68)$$

Proof. The Theorem 6.11 of [KLT97] tells that the element $(1-q)\Psi_n(\frac{A}{1-q})$ is

$$\frac{1}{[n]_q}\sum_{|I|=n}(-1)^{d(I)}\begin{bmatrix} n-1 \\ d(I)\end{bmatrix}_q^{-1} q^{\mathrm{maj}(I)-\binom{d(I)+1}{2}}\, R_I. \qquad (69)$$

By Prop. 5.9, the image by θ of this formula is $(-1)^{n-1}\varphi(\Omega_{q,n})$. By Prop. 5.8, this becomes the expected formula. ∎

6 Arithmetic properties

In this section, we obtain some properties of the denominators in Ω_q and consider what happens when q is specialized to $1, 0$ and ∞.

6.1 $q = 1$

Let us first note that the morphism φ from $\widehat{\mathsf{PL}}$ to the completed free dendriform algebra $\widehat{\mathrm{Dend}}$ is defined over $\mathbb{Q}$ and injective. Hence one can deduce results on Ω_q from results on its image by φ.

Proposition 6.1 *The series Ω_q is regular at $q=1$ and $\Omega_{q=1}=\Omega$.*

Proof. By Proposition 5.10, the image $\varphi(\Omega_q)$ is regular at $q=1$, as q-binomial coefficients become usual binomial coefficients when $q=1$. Therefore Ω_q itself is regular at $q=1$.

At $q=1$, the equation (19) becomes the equation (12). By uniqueness in Proposition 3.3, the value of Ω_q at $q=1$ is Ω. ∎

Remark: knowing that $\Omega_{q=1}=\Omega$, one can use equation (20) to compute simultaneously Ω_q and Ω up to order n in a $O(n^3)$ number of pre-Lie operations.

There is a lot of cancellations in the coefficients of Ω_q, leading to a reduced complexity of the denominators. Note that the expected denominator of $\Omega_{q,n}$ (from recursion (20)) is the product $\prod_{d=2}^n(q^d-1)$. Let Φ_d be the d^{th} cyclotomic polynomial.

Proposition 6.2 *The common denominator of the coefficients of the element $\Omega_{q,n}$ divides the product $\prod_{d=2}^n\Phi_d$.*

Proof. For the image of Ω_q by φ, this follows from Prop. 5.10 and a simple property of the q-binomial coefficients: their only roots are simple roots at roots of unity, see [GZ06, Prop 2.2]. This implies the same result for Ω_q. ∎

6.2 $q = 0$

Let us consider now what happens when $q = 0$. Then Ω_0 is well-defined, $\Omega_q[q]$ vanishes and the equation (19) becomes

$$\Omega_0 = \bullet - \bullet \curvearrowleft \Omega_0. \tag{70}$$

It follows that Ω_0 is the alternating sum of linear trees.

6.3 $q = \infty$

Let us now consider what happens when $q = \infty$. Let $\omega_{q,T}$ be the coefficient of the rooted tree T in the expansion of Ω_q. We will call **valuation** at $q = \infty$ the smallest exponent in the formal Laurent expansion in powers of q^{-1} of an element of $\mathbb{Q}(q)$.

Proposition 6.3 *The valuation of $\omega_{q,T}$ at $q = \infty$ is at least $\#T - 1$.*

Proof. This will follow from the recursion (20). This is true in degree $n = 1$.

Let us assume that $n \geq 2$. Then the valuation of $\bullet \curvearrowleft \Omega_{q,n-1}$ is at least $n-2$ by induction and the valuation of each term of the rightmost sum in equation (20) is at least -1. Hence the valuation of $\Omega_{q,n}$ is at least $n-1$. ■

Hence there exists a limit Ω_∞ for $\Omega_q[q]/q$ when q goes to ∞ and the limit of Ω_q/q is zero.

The equation (19), divided by q, becomes at $q = \infty$,

$$\Omega_\infty \curvearrowleft \exp(\Omega) = \bullet. \tag{71}$$

By right action by $\exp(-\Omega)$, this is equivalent to

$$\Omega_\infty = \bullet \curvearrowleft \exp(-\Omega). \tag{72}$$

The element $\exp(-\Omega)$ is the inverse of $\exp(\Omega)$ in $\widehat{U}(\mathsf{PL})$. This has been computed in [CL07, §6.4]. More precisely, the inverse of $\sum_{n\geq 1} \frac{1}{(n-1)!}\mathtt{Crl}_n$ in the group of characters of the Connes-Kreimer Hopf algebra was shown there to be

$$\sum_T \frac{(-1)^{\#T-1}}{\operatorname{aut}(T)} T, \tag{73}$$

where $\operatorname{aut}(T)$ is the cardinal of the automorphism group of the rooted tree T. But it is known [CL01] that this group of characters is isomorphic to the group of group-like elements in $\widehat{U}(\mathsf{PL})$. Going through the isomorphism, one gets the following result.

Proposition 6.4 *The series Ω_∞ is given by*

$$\Omega_\infty = \sum_T \frac{(-1)^{\#T-1}}{\operatorname{aut}(T)} T. \tag{74}$$

7 Morphisms and images

In this section, we consider two quotients of the free pre-Lie algebra PL and the images of Ω_q in these quotients. We will use some results of [Cha02a].

7.1 Morphism to the free associative algebra

Consider the free (non-unital) associative algebra on one generator x, denoted by $\mathbb{Q}[x]_+$. As the associative product is also a pre-Lie product, there exists a unique morphism of pre-Lie algebras from PL to $\mathbb{Q}[x]_+$ sending $\bullet$ to x. This extends uniquely to a morphism from $\widehat{\mathsf{PL}}$ to the algebra $\mathbb{Q}[[x]]_+$ of formal power series in x without constant term.

One can show that this morphism send the linear trees $\mathtt{Lnr}_n$ with n vertices to the monomials x^n and all others trees to 0.

It is known (see [Cha02a]) that the image of Ω is the formal power series

$$\log(1+x) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} x^n. \tag{75}$$

Therefore the image of $\exp(\Omega)-1$ is just x. Note that the right action is mapped to the product.

One deduces from Eq. (19) that the image of Ω_q is the q-logarithm defined by

$$\log_q(x) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{[n]_q} x^n, \tag{76}$$

which is the unique solution to the functional equation

$$x \log_q(qx) = x \log_q(x) + (q-1)\, x - \log_q(qx) + \log_q(x). \tag{77}$$

7.2 Morphism for corollas

As shown in [Cha02a], the subspace of PL spanned by trees that are not corollas is a two-sided pre-Lie ideal.

The quotient pre-Lie algebra is isomorphic to the following pre-Lie algebra. Let us identify the image of the corolla $\mathtt{Crl}_{n+1}$ with n leaves to x^n for all $n \geq 0$. In particular, the tree $\bullet$ is mapped to 1. The underlying vector space is therefore identified with $\mathbb{Q}[x]$ and the pre-Lie product is

$$x^p \curvearrowleft x^q = \begin{cases} x^{p+1} & \text{if} q = 0, \\ 0 & \text{else.} \end{cases} \tag{78}$$

It is known (see [Cha02a]) that the image of Ω is the generating function $\frac{x}{\exp(x)-1}$ for the Bernoulli numbers.

One can also show (using the description of the quotient $\curvearrowleft$ product given above) that the right action of the image of $\exp(\Omega)-1$ is just given by the product by $\exp(x)-1$ and the right action by the image of Ω_q is given by the product by x.

Then, from Eq. (19), one gets that the image $F_q(x)$ of Ω_q satisfies the following equation

$$(\exp(x)-1)[qF_q(qx)] = x + q - 1 - qF_q(qx) + F_q(x). \tag{79}$$

This functional equation is known (see for instance [Sat89]) to describe the generating function

$$F = \sum_{n\geq 0} \beta_n(q) \frac{x^n}{n!}, \tag{80}$$

where $\beta_q(n)$ are the q-Bernoulli numbers introduced by Carlitz in 1948, see [Car48, Car54, Car58].

Therefore the coefficients of the corollas in Ω_q are the q-Bernoulli numbers of Carlitz.

7.3 Morphism to a pre-Lie algebra of vector fields

There exists an interesting morphism from PL to a pre-Lie algebra of vector fields. We describe it here only as a side remark, as the image of Ω_q seems to have no special property.

Consider the the vector space $V = \mathbb{Q}[x]_+$, endowed with the following pre-Lie product:

$$(f \curvearrowleft g) = xf\,\partial_x g. \tag{81}$$

Then there is a unique morphism from PL to V sending $\bullet$ to x.

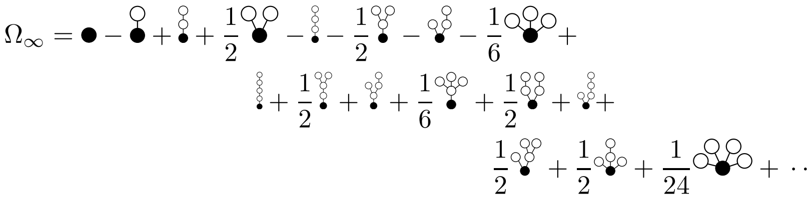
This map has the following nice property: the coefficient of x^n in the image of a series A is the sum of the coefficients of the trees in the homogeneous component A_n of A. The proof is just a check that this sum-of-coefficients map defines a morphism of pre-Lie algebra from PL to V.

8 First terms of some expansions

$$\Omega = \bullet - \frac{1}{2} + \frac{1}{3} + \frac{1}{12} - \frac{1}{4} - \frac{1}{12} - \frac{1}{12} + \frac{1}{5} + \frac{3}{40} + \frac{1}{10} + \frac{1}{180} + \frac{1}{60} + \frac{1}{20} + \frac{1}{120} - \frac{1}{120} - \frac{1}{720} + \cdots \tag{82}$$

For all $n \geq 1$, let Φ_n be the n^{th} cyclotomic polynomial.

$$\Omega_q = \bullet - \frac{1}{\Phi_2} + \frac{1}{\Phi_3} + \frac{q}{2\,\Phi_2\Phi_3} - \frac{1}{\Phi_2\Phi_4} - \frac{q}{2\,\Phi_3\Phi_4} - \frac{q^2}{\Phi_2\Phi_3\Phi_4} - \frac{q(q-1)}{6\,\Phi_2\Phi_3\Phi_4} + \frac{1}{\Phi_5} + \frac{q(1+q+q^2)}{2\,\Phi_2\Phi_4\Phi_5} + \frac{q^2}{\Phi_4\Phi_5} + \frac{q(q^3+q^2-1)}{6\,\Phi_3\Phi_4\Phi_5} + \frac{q^4}{2\,\Phi_3\Phi_4\Phi_5} + \frac{q^3}{\Phi_2\Phi_4\Phi_5} + \frac{q^2(q^3+q^2-1)}{2\,\Phi_2\Phi_3\Phi_4\Phi_5} + \frac{q^2(q^3-q-1)}{2\,\Phi_2\Phi_3\Phi_4\Phi_5} + \frac{q(q^4-q^3-2q^2-q+1)}{24\,\Phi_2\Phi_3\Phi_4\Phi_5} + \cdots \tag{83}$$



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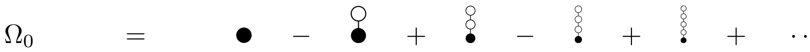

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